

Exact N - vortex solutions to the Ginzburg - Landau equations for $\kappa = 1/\sqrt{2}$

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(February 1, 2008)

Abstract

The N-vortex solutions to the two-dimensional Ginzburg - Landau equations for the $\kappa = 1/\sqrt{2}$ parameter are built. The exact solutions are derived for the vortices with large numbers of the magnetic flux quanta. The size of vortex core is supposed to be much greater than the magnetic field penetration depth. In this limiting case the problem is reduced to the determination of vortex core shape. The corresponding nonlinear boundary problem is solved by means of the methods of the theory of analytic functions.

74.20.De, 74.60.Ec, 02.30.Jr

I. INTRODUCTION

In different areas of physics there is a wide interest in nonlinear field equations admitting particlelike solutions. The Ginzburg-Landau (GL) theory [1] is a well known example of this kind. In the case of a hard superconductor subjected to an external magnetic field the GL equations give vortex-line solutions. The properties of the vortex states in the GL theory are investigated in two limiting cases [2]. In the first case the sector of the fields near the lower critical field H_{c_1} is considered. In this sector a mixed state of the type-II superconductor is a system of widely separated vortices. In this case the problem is solved for the unit vortex. Due to the cylindrical symmetry of the system the corresponding GL equations are reduced to the set of the ordinary differential equations. In the second case the sector of the fields near the upper critical field H_{c_2} is considered. In this region the periodical lattice of the vortices is the equilibrium state of the superconductor. Due to the small value of the order parameter the GL equations turn out to be linear. It allows one to investigate them by the usual methods.

In this work the exact solutions of nonlinear GL equations are built. The solutions are derived in the special case of the $\kappa = \delta/\xi = 1/\sqrt{2}$, where δ is the magnetic field penetration depth, ξ is the fluctuation radius of the order parameter, and in the asymptotic of the large magnetic fluxes of each vortex $\Phi \gg \Phi_0$, where $\Phi_0 = \pi\hbar c/e$ is the magnetic flux quantum. In the limiting case when the vortex core sizes considerably exceed the typical lengths δ or ξ the problem is reduced to the purely geometrical one, to the determination of the shape of the contour confining the vortex cores. Such a problem proves to be nonlinear. Nevertheless, it is solved by means of methods of the theory of analytic functions.

This case is interesting since the quantum properties of the superconductor are manifested in the classical limit. The κ parameter determines the value of tension coefficient on the interphase of normal and superconducting phase. It is at $\kappa = 1/\sqrt{2}$ that the coefficient vanishes [3] and causes the agreement of all three critical magnetic fields $H_{c_1} = H_{c_2} = H_c$. Therefore, the material with such value of the κ parameter possesses the properties of both

type-I and type-II superconductors. On the one hand, the vortices with large magnetic fluxes are the incorporations of the normal phase. They should be considered as the parts of the intermediate state of type-I superconductor. On the other hand, such regions of the normal phase are nothing but the vortices. As shown below their shape depends on the coordinates of the zeros of the order parameter and the numbers of the magnetic flux quanta in every vortex, i.e. on the typical parameters of the mixed state.

The vortex states with large magnetic fluxes under consideration are not too "exotic" objects. Parallel with the macroscopic vortices the quantum vortices can also exist in superconductor. This is caused by the absence of the interaction among the vortices. The energy of the whole system depends only on the total magnetic flux [4]. At the given total magnetic flux the decay of the large vortices into small vortices does not lead to a gain in energy.

This consideration is based on the results of Ref. [4]. In this work it is shown that the GL equations are reduced to a set of the first order coupled equations for the vector potential and the order parameter. By appropriate choice of the variables the latter are written in the form of an equation similar to the Poisson equation [5]. Its form is exactly the one that has the asymptotically exact solutions [6].

II. GENERAL APPROACH

Let us consider the two-dimensional distributions of the complex order parameter $\psi(\mathbf{r})$ and the vector potential $\mathbf{A}(\mathbf{r})$. Such distributions are realized for the superconducting cylinder subjected to the longitudinal magnetic field (see Fig. 1). If no edge effects are present all, values depend on the transversal coordinates x and y only.

After rescaling the lengths and the fields the free energy of superconductor per unit length of the sample is written as follows

$$F = \int d^2\mathbf{r} \left[\frac{1}{2} \mathbf{B}^2 + \frac{1}{2} |(\nabla - i\mathbf{A}) \psi|^2 \right]$$

$$+ \frac{\lambda^2}{8} (|\psi|^2 - 1)^2 \Big], \quad (1)$$

where all the lengths are measured in the units of magnetic penetration depth δ , and $\lambda = \sqrt{2}\kappa$ is the dimensionless coupling constant. For the chosen variables the physical magnetic field $B = H_c$ corresponds to the dimensionless field $B = \lambda/2$.

It is convenient to introduce the complex variables such as

$$z = x + iy, \quad \bar{z} = x - iy \quad (2)$$

and the vector potential

$$A = \frac{1}{2} (A_x - iA_y), \quad \bar{A} = \frac{1}{2} (A_x + iA_y). \quad (3)$$

These notations us permit to present the equations derived in Ref. [4,5] for $\kappa = 1/\sqrt{2}$ and, correspondingly, $\lambda = 1$ in the most simple form. Such equations give the minimum of free energy (1). Unlike the GL equations they are the first order with respect the derivatives:

$$\frac{\partial \psi}{\partial \bar{z}} - i\bar{A}\psi = 0, \quad (4)$$

$$B + \frac{1}{2} (|\psi|^2 - 1) = 0. \quad (5)$$

These equations give the vortexlike solutions with the magnetic flux

$$\Phi = \int d^2r B = 2\pi n, \quad (6)$$

where n is the positive integer number. The corresponding total free energy of the system in dimensionless units turns out to be equal exactly to πn .

The set of the coupled equations (4) and (5) is reduced to one equation by the substitution of the form [5]:

$$A = -\frac{i}{2} \frac{\partial}{\partial z} (u + 2 \ln |f|), \quad (7)$$

$$\psi = \frac{f}{|f|} \exp \left(-\frac{u}{2} \right), \quad (8)$$

where $u(x, y)$ is the real-valued function, and $f(z)$ is the analytic function. After the substitution the former equation of the system becomes the identity. The second equation takes the form

$$\begin{aligned}\Delta u &= -4\pi\rho(u) - 2\Delta \ln |f| \\ &= -4\pi\rho(u) - 4\pi \sum_{j=1}^N n_j \delta(\mathbf{r} - \mathbf{r}_j),\end{aligned}\tag{9}$$

where Δ is the Laplace two-dimensional operator,

$$\rho(u) = \frac{1}{4\pi} [\exp(-u) - 1].\tag{10}$$

Formula (9) differs from the analogous one of Ref. [5] in the last term of the right-hand side. Here singularities of the function $\ln |f(z)|$ are taken into account. The analytic function $f(z)$ vanishes like $(z - z_j)^{n_j}$ at the positions of the vortices z_j , where n_j is the positive integer number, $j = 1, \dots, N$, and N is the number of the vortices [5,7]. This leads to the δ functions in the right-hand side of the equation (9).

The boundary condition for this equation is vanishing of the value $u(x, y)$ at the infinity. According to expression (8) this requirement gives equality $|\psi| = 1$. The function $f(z)$ is to some degree an arbitrary one. Its form depends on the choice of the gauge of the vector potential.

The derived equations have the following physical meaning. They evidently describe the screening of the external positive point charges in a classical plasma. The value of the $u(\mathbf{r})$ is the electrostatic potential of the system of charges. The dependence of $\rho(u)$ gives the induced charge density described by the jellium model which consists of a negative smeared background and mobil charged fluid.

The exact solutions of nonlinear screening equation (9) are built in the limiting case $n_j \gg 1$. In this approximation a sufficiently large positive potential is supposed to exist near external point charges. As a consequence the screening results from a complete repulsion of the positively charged fluid from the regions surrounding the point charges. The induced charge density turns out to be approximately constant within these regions and is equal to

$\rho(\infty) = -1/4\pi$. The size of such region is much greater than the Debye length for linear screening.

In these assumptions one can ignore the details of the potential behavior on the scale of the order of the Debye length. Namely, one can suppose that the system consists only of the uniformly charged spatial region and a set of point charges. In this case the screening of the point charges can be achieved by means of selecting such a form of the region in which everywhere outside the region both the potential and the electric field would vanish. This problem is much easier than the initial one. Due to its two-dimensionality it is solved by the methods of the theory of functions of a complex variable [6].

In the given case the results of the work [6] can be rewritten in the following form. The solution is given by the function

$$z = \omega(\zeta) = c_0 + \sum_{j=1}^N \frac{c_j}{\zeta - \zeta_j}, \quad (11)$$

which makes a conformal mapping of the inner region of the unit circle $|\zeta| \leq 1$ in an auxiliary plane of the complex variable ζ into the region of interest. The constants ζ_j obey the condition $|\zeta_j| > 1$ for all j . The shape of the contour is specified parametrically: $z = \omega(\zeta)$ for $\zeta = \exp(i\alpha)$ and $0 \leq \alpha \leq 2\pi$.

The set of unknown constants c_j and ζ_j is calculated by means of the system of nonlinear algebraic equations

$$\begin{aligned} \omega(\xi_j) &= z_j, \\ \bar{c}_j \xi_j^2 \omega'(\xi_j) &= -4n_j, \end{aligned} \quad (12)$$

where $\xi_j = 1/\bar{\zeta}_j$, $j=1, \dots, N$; $\omega'(\zeta)$ is the derivative of the function $\omega(\zeta)$. Constant c_0 is determined from the additional requirement that an inner point of the unit circle should correspond to a given point within the uniformly charged region.

The potential $u(x, y)$ is given by the expression

$$u(x, y) = \frac{1}{4} |z|^2 - 2\text{Re } W(z), \quad (13)$$

where

$$W(z) = \frac{1}{8}v(z) + \sum_{j=1}^N n_j \ln \left[\frac{\bar{\zeta}_j \zeta(z) - 1}{\zeta(z) - \zeta_j} \right], \quad (14)$$

$$v(z) = |c_0|^2 + \sum_{j,k=1}^N \frac{\bar{c}_j c_k}{1 - \bar{\zeta}_j \zeta_k} + 2 \sum_{j=1}^N \frac{\bar{z}_j c_j}{\zeta(z) - \zeta_j}.$$

Here function $\zeta(z)$ is the inverse of $z = \omega(\zeta)$.

The results of the electrostatic problem allow one to write the corresponding solutions of the GL equations easily. In the next section the general formulas are given and some simple cases are considered.

III. RESULTS AND DISCUSSION

Taking into account the fact that the gauge of the vector potential can be an arbitrary one, let us take the function $f(z)$ in the simplest form

$$f(z) = \prod_{j=1}^N (z - z_j)^{n_j}. \quad (15)$$

Within the core of the vortices the order parameter and the vector potential will be described by the expressions:

$$\psi = \prod_{j=1}^N \left[\frac{z - z_j}{|z - z_j|} \left| \frac{\bar{\zeta}_j \zeta(z) - 1}{\zeta(z) - \zeta_j} \right| \right]^{n_j} \times \exp \left(-\frac{|z|^2 - \text{Re } v(z)}{8} \right), \quad (16)$$

$$A = -\frac{i}{8} \left[\bar{z} - \eta(z) + 4 \sum_{j=1}^N \frac{n_j}{z - z_j} \right], \quad (17)$$

where

$$\eta(z) = \bar{c}_0 + \sum_{j=1}^N \frac{\bar{c}_j \zeta(z)}{1 - \bar{\zeta}_j \zeta(z)}. \quad (18)$$

In formula (17) value A will be regular near the points $z = z_j$, because the singularities of the function $\eta(z)$ are eliminated by the third term in square brackets.

Outside the vortex cores the order parameter and the vector potential are described by

$$\psi = \prod_{j=1}^N \left(\frac{z - z_j}{|z - z_j|} \right)^{n_j}, \quad (19)$$

$$A = -\frac{i}{2} \sum_{j=1}^N \frac{n_j}{z - z_j}. \quad (20)$$

In this region the induction of magnetic field B equals zero. On the contrary, within the vortex core the induction is $B = 1/2$. The last value corresponds to the field $B = H_c$ measured in the physical units.

Formulas (16)-(20) give the solutions only when the vortex cores are connected to each other [6]. For the sets of widely separated vortices these formulas should be applied to each connected group. These expressions do not work for the regions of "Swiss cheese" type. A solution of an electrostatic problem of this kind was obtained in Ref. [8].

Formulas (16)-(20) present the desired results as the functions of the auxiliary variable ζ . To obtain the explicit dependence of these functions on the variable z it is necessary to invert the equality $z = \omega(\zeta)$ with respect to the variable ζ . It is possible only for some vortices.

Let us consider two special cases. The simplest one is the problem for the unit vortex in the origin of the coordinates $z = 0$. The corresponding solutions have the form

$$\begin{aligned} \psi &= \left(\frac{z}{r} \right)^n \exp \left[-\frac{1}{8} (|z|^2 - r^2) \right], \\ A &= -\frac{i}{8} \bar{z}, \end{aligned} \quad (21)$$

where $|z| \leq r$. The vortex core is a circle with the radius $r = 2\sqrt{n}$, where n is the number of the flux quanta of the magnetic field.

The problem for the twin vortices placed at the points $x = \pm a$ of the real axes can be regarded as the second example. The vortex cores are overlapped if the distance a obeys the inequality $a \leq r$. Their shape is described by the conformal mapping function

$$z = \omega(\zeta) = \frac{2c_1\zeta}{\zeta_1^2 - \zeta^2}, \quad (22)$$

where

$$\begin{aligned}
c_1 &= a\sqrt{(p^2 - 1) \left(p + \sqrt{p^2 - 1}\right)}, \\
\zeta_1 &= \sqrt{p + \sqrt{p^2 - 1}}, \\
p &= \left(\frac{r}{a}\right)^2.
\end{aligned} \tag{23}$$

The order parameter is given by the expression

$$\begin{aligned}
\psi &= \left(\frac{z^2 - a^2}{a \left| r + \sqrt{z^2 - a^2 + r^2} \right|} \right)^n \\
&\times \exp \left[-\frac{|z|^2}{8} + \frac{a}{4} \operatorname{Re} \left(\sqrt{z^2 - a^2 + r^2} \right) \right].
\end{aligned} \tag{24}$$

The corresponding formula for the vector potential is too bulky to be presented here.

Note that solutions (16)-(17) may be rewritten in a form similar to the eigenfunction of a charged particle in the lowest Landau level:

$$\psi = F(z) \exp \left(-\frac{|z|^2}{8} \right), \tag{25}$$

where $F(z)$ is an analytic function. Such a presentation is evident for the following physical reasons. Within the normal phase the magnetic-field induction is approximately constant and equal to the critical field H_c . The order parameter is negligibly small here as well. The GL equations linearized with respect to the smallness $|\psi| \ll 1$ really give such solutions.

To derive the presentation (25) let us take function $f(z)$ in the form $f(z) = \exp(W(z))$. Correspondingly, formula (13) gives the equality $F(z) = f(z)$. The vector potential takes the form $A = -i\bar{z}/8$.

In conclusion, let us discuss the validity of the model. It was assumed above that the order parameter depends on two coordinates only. This distribution is not realized in fact. According to the theory of the intermediate state of superconductors the domains of a normal phase decay near the surface of a sample into the alternating layers of normal and superconducting phases [9]. Thus, there is a dependence on the third coordinate directed along the external magnetic field. In this connection the existence of the macroscopic vortices may be provided only by a special mechanism for the attraction of vortices. The pinning

of vortices near space inhomogeneities or free boundary of the sample can become such a mechanism. The solution of this problem is beyond the scope of this article.

ACKNOWLEDGMENTS

This work was supported by Volkswagen Stiftung Foundation Grant No. I/68553. The author is grateful to E.G.Batyev and M.V.Entin for useful discussions.

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FIGURES

FIG. 1. The geometry of the system. The shaded region of the picture shows a unit vortex in a cylindrical specimen of superconductor.